Geometric problems related to the Euler Characteristic Transform

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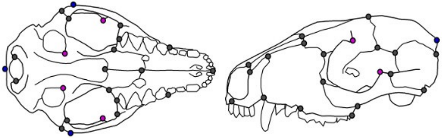
May 4 2024

Outline

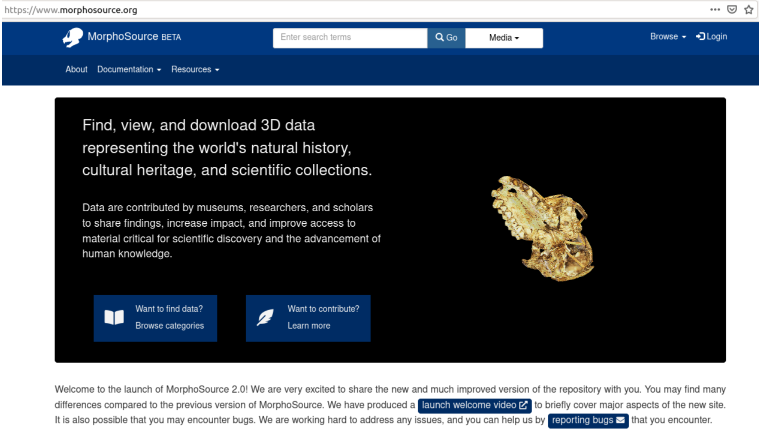
- Scientific Background
- The Euler Characteristic Transform
- Digitalization and problems

Classic Shape Theory (Kendall)

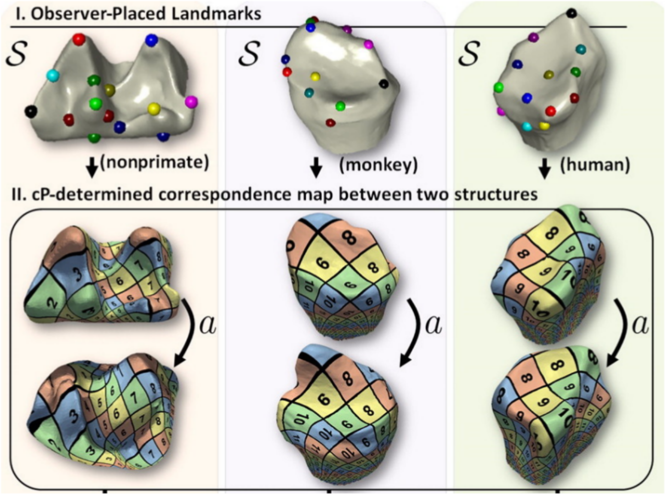


$$d(\mathcal{P}, \mathcal{Q}) = \frac{1}{n} \inf_{R \in \mathcal{R}} \sum_{i=1}^{n} ||p_i - Rq_i||^2$$

Digital Museums

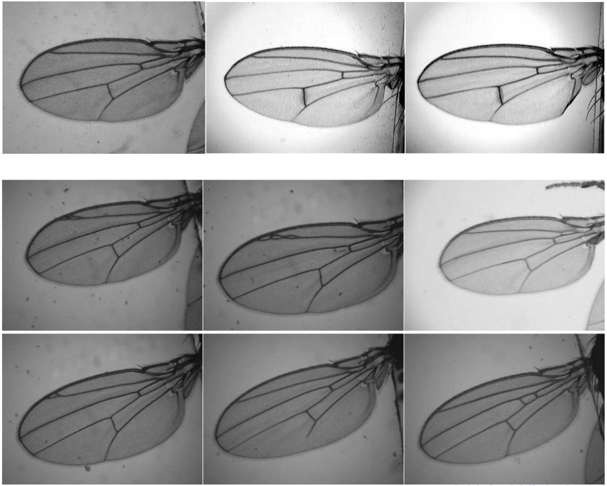


Defenders of Geometric Morphometrics



Non-diffeomorphic shapes

Images courtesy of David Houle's laboratory



Differential Geometry $\hookrightarrow$ Geometric Measure Theory

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- Valuations on $\mathbb{R}^d$
  - $\Psi(A \cup B) + \Psi(A \cap B) = \Psi(A) + \Psi(B)$
  - Continuous and invariant under rotations and translations

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Theorem (Hadwiger '57)

The set of such valuations can be thought of as a vector space of polynomials in scale x, i.e. there is a basis $1, x, x^2, x^3, \ldots, x^d$.

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Theorem (Hadwiger '57)

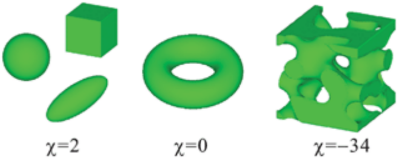
The set of such valuations can be thought of as a vector space of polynomials in scale x, i.e. there is a basis $1, x, x^2, x^3, \ldots, x^d$.

- There exists a canonical scale-free valuation: the 0-homogeneous intrinsic volume, also known as the *Euler Characteristic*

Euler Characteristic

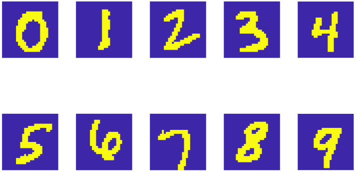
Definition

The *Euler characteristics* χ of a convex compact set and the empty set are 1 and 0, respectively. For other tame sets



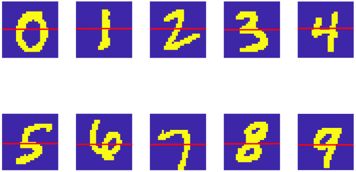
$$\chi(A \cup B) + \chi(A \cap B) = \chi(A) + \chi(B)$$

Euler Characteristic



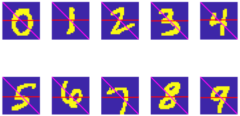
X	$\chi(X)$
0	0
1	1
2	1
3	1
4	1
5	1
6	0
7	1
8	-1
9	0

Euler Characteristic



Shape	χ	$\chi(+)$	$\chi(-)$
0	0	1	1
1	1	1	1
2	1	1	1
3	1	1	1
4	1	2	1
5	1	1	1
6	0	2	1
7	1	1	1
8	-1	0	0
9	0	0	1

Euler Characteristic



Shape	χ	$\chi(+)$	$\chi(-)$	$\chi(+)$	$\chi(-)$
0	0	1	1	1	1
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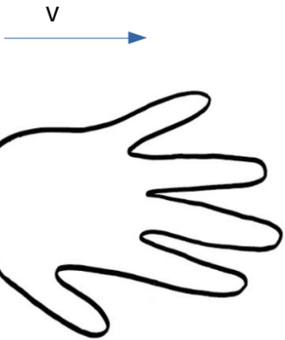
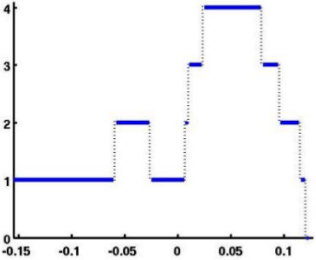
2	1	1	1	2	2
3	1	1	1	1	3
4	1	2	1	2	1
5	1	1	1	2	1
6	0	2	1	2	1
7	1	1	1	1	2
8	-1	0	0	0	0
9	0	0	1	1	1

Euler Curves

For a shape $X \subset \mathbb{R}^d$ and direction $v \in S^{d-1}$, the *Euler Curve* of X in direction v:

$$\mathrm{ec}_{(X,v)}(h) : \chi(\{x \in X | \langle x, v\rangle \leq h\})$$





ECT - Good properties

Look at all the directions and heights simultaneously:

$$X \hookrightarrow \mathbf{ECT}_X(v,h) : S^{d-1} \times \mathbb{R} : (v,h) \mapsto \chi(\{x \in X | x \cdot v \leq h\})$$

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- Injective
- Invertible Transform

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- Inner products
- Injective
- Invertible Transform
- $SO(d)$-Equivariant: $ECT_{AX}(v,t) = ECT_X(A^{-1}v,t)$.

ECT in Practice

Discretize $S^{d-1} \times \mathbb{R}$ and evaluate it on a grid. But this is not digital! More problems:

- Inner products - No problem
- Injective - Practically yes, and theoretical results exists. Terms and conditions apply
- Invertible Transform - Locally possible
- Equivariance: Expensive and approximate
- Finer discretization: Better representation? Multicollinearity
- Choice of discretization? Research dissemination

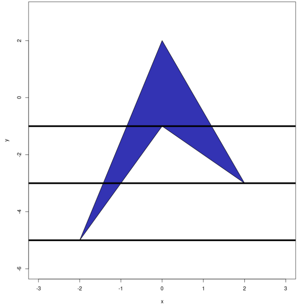
Discretization related problems

- How many directions are needed?[1]
- How to tell if a given discretization came from a valid shape?
- (The above especially relevant for the PHT, where we record the betti numbers instead of euler characteristic)

[1] Partial answers exist

Towards Digitalization

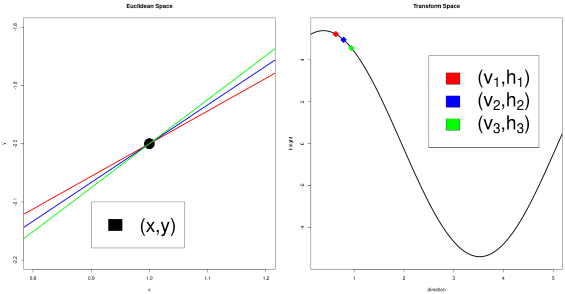
X piecewise linear.
Observation 1. For any direction v, the euler curve $\mathrm{ec}_{X,v}(h)$ can only jump at vertices x_i of X ($h = v \cdot x_i$) for some i.



$h_1 = (0,1) \cdot (-2,-5) = -5$
$h_2 = (0,1) \cdot (2,-3) = -3$
$h_3 = (0,1) \cdot (0,-1) = -1$
$h_4 = (0,1) \cdot (0,2) = 2$ (no jump).

Towards Digitalization

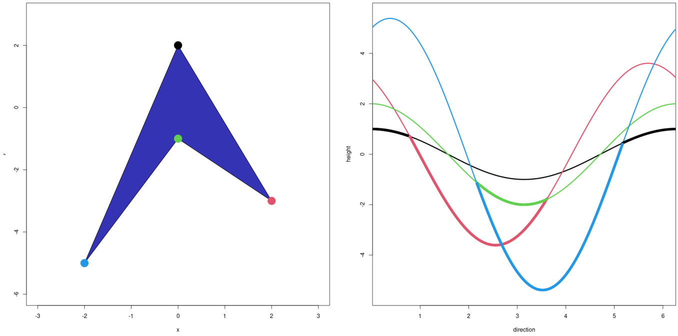
Observation 2. The height function of a vertex $p_i = (r, \phi)$ is given by $h_i = r\cos(\theta - \phi)$.



$$\begin{pmatrix} \mathbf{v_1} \\ \mathbf{v_2} \\ \mathbf{v_3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

Towards Digitalization

Observation 3.
The height functions divide $S^{d-1} \times \mathbb{R}$ up into strata where the ECT is constant.

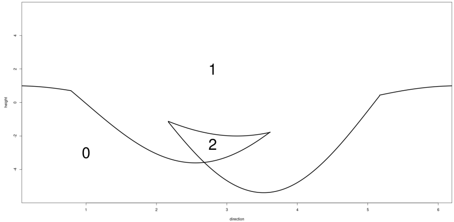


The Brute Force algorithm (2d)

- Solve $< v, x_i > = < v, x_j >$ for all $i \neq j$, get $v_{i,j}, -v_{i,j}$
- Order the $v_{i,j}$ by their angle, get pieces $P_k = [\theta_k, \theta_{k+1}]$, $k = 1, \ldots, N$.
- Set $\alpha_k = \frac{\theta_k + \theta_{k+1}}{2}$
- Look at the order of x_i under $< \alpha_k, x_i >$.
- For each x_i, evaluate the discrete ECT at height $h_i \pm$, i.e. just above and below x_i, if non-zero, record this to $a_{k,i}$ and record x_i to $b_{k,i}$.
- The ECT of X is now represented as $\bigcup_{k=1}^{N} (P_k, \mathbf{a}_k, \mathbf{b}_k)$.

Towards Digitalization

Observation 4. The behavior of the euler curve at h_i depends only on the neighborhood of x_i. (Note same is not true for PHT)



- Procedure can be parallelized over neighborhoods of points.
- Reduce intersection from $O(n^d)$ to $O(n)$ (note d typically 2 or 3)

ECT Inner products

$$ECT_X(v,h) = \sum_{i=1}^{N} a_i \mathbb{1}_{\geq b_i}(h) \mathbb{1}_{[\theta_i, \phi_i]}(v),$$

i.e. a simple function of indicators in h and v.
The inner product is just the L_2 inner product:

$$\langle ECT_X, ECT_Y \rangle = \int_{S^{d-1}} \int_{-1}^{1} ECT_X ECT_Y \, dh \, dv$$

NB: Product of two simple functions is still simple.

Integrating over the sphere

Let $p_i = (x_i, y_i)$ be a vertex. The integral of the height function of p_i from τ to θ is given by:

$$I(p_i) = \int_\tau^\theta \cos(t)x_i + \sin(t)y_i \, dt$$
$$= \Big(\cos(\tau) - \cos(\theta)\Big)y_i + \Big(\sin(\theta) - \sin(\tau)\Big)x_i.$$

Similar explicit formula exists also for 3D (HK, Xiaohan Wang)

Digital Advantage I: Schapira's inversion

$$(\mathcal{R}_{k'} \circ \mathcal{R}_k)h = (\mu - \lambda)h + \int_X h \, d\chi \mathbb{1}.$$

Concretely for a 2d shape:

$$\mathbb{1}_X(x) = \int_{x \cdot v = h} \mathbf{ECT}(v, h) + \mathbf{ECT}(-v, -h)\chi(dv, dt) - \chi(X).$$

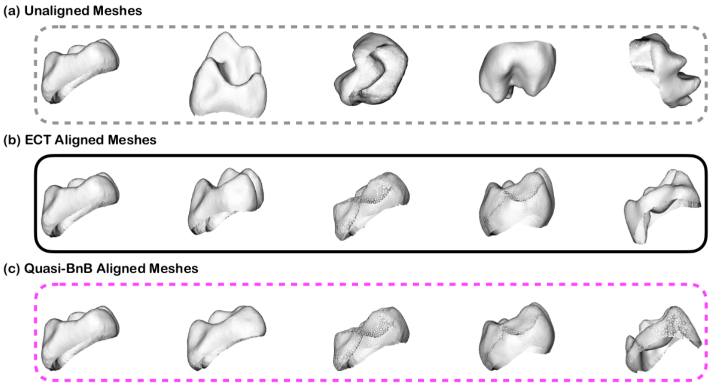
Digital Advantage II: Genuine $SO(d)$ action

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{ECT}(X) \\ \Big\downarrow \phi & & \Big\downarrow \phi \\ Y & \longrightarrow & \mathrm{ECT}(Y) \end{array}$$

$$\begin{aligned} &\langle \mathrm{ECT}(X_1) - \mathrm{ECT}(\phi X_2), \mathrm{ECT}(X_1) - \mathrm{ECT}(\phi X_2)\rangle \\ &= \langle \mathrm{ECT}(X_1) - \phi\mathrm{ECT}(X_2), \mathrm{ECT}(X_1) - \phi\mathrm{ECT}(X_2)\rangle \\ &= \langle \mathrm{ECT}(X_1), \mathrm{ECT}(X_1)\rangle - 2\langle \mathrm{ECT}(X_1), \phi\mathrm{ECT}(X_2)\rangle \\ &+ \langle \phi\mathrm{ECT}(X_2), \phi\mathrm{ECT}(X_2)\rangle \\ &\propto -\langle \mathrm{ECT}(X_1), \phi\mathrm{ECT}(X_2)\rangle. \end{aligned}$$

The map $O(3) \to \mathbb{R} : \phi \mapsto \langle \mathrm{ECT}(X_1), \phi\mathrm{ECT}(X_2)\rangle$ is almost differentiable.

Example of aligning shapes with ECT



n=2,918, N=16,000

Summary of the digitalization

+ Uses all information in the data
+ No need for extra parameters / non-canonical choices
+ Opens doors for more involved questions
- Complicated algorithm
- Expensive to compute
- No partial progress

Question I

Given n points $X = \{x_1, \ldots, x_n\}$ in $\mathbb{R}^d$, define an equivalence relation on linear functions $f : \mathbb{R}^d \to \mathbb{R}$ that are injective on X given by the order of the vertices x_i.

I a For a point cloud X, how many equivalence classes are there?

I b For a fixed d and n, which point configuration attains the maximum?

Question II

An effective way to sort lists is the mergesort, which is a sort of divide and conquer algorithm.
In 2d, it is fairly easy to devise a divide and conquer algorithm that speeds the intersection checks exponentially over brute force.

- What is an effective way to check intersection of collection of triangles on S^2 (or S^d in general)

Question III

Given a shape $X \subset \mathbb{R}^d$, define it's sinogram complexity $S(X)$ as the number of d-dimensional strata in $S^{d-1} \times \mathbb{R}$ its Euler characteristic transform.

- III a What are the properties of $S(X)$?
- III b Give a characterization of shapes of given sinogram complexity.
- III c Give a fast to compute algorithm to approximate the sinogram complexity of a shape
- III d Define a better notion of shape complexity that behaves better under e.g. set union
- III e (Try any of the above with the PHT)

For example, $S(X) = 2$ is the set of convex bodies.

Vielen Dank für Ihre Zeit